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K-THEORY OF FUNCTION RINGS

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The ring R of continuous functions on a compact topological space X with values in $\mathbb{R}$ or $\mathbb{C}$ is considered. It is shown that the algebraic K-theory of such rings with coefficients in $\mathbb{Z}/k\mathbb{Z}$, k any positive integer, agrees with the topological K-theory of the underlying space X with the same coefficient rings. The proof is based on the result that the map from R^δ (R with discrete topology) to R (R with compact-open topology) induces a natural isomorphism between the homologies with coefficients in $\mathbb{Z}/k\mathbb{Z}$ of the classifying spaces of the respective infinite general linear groups. Some remarks on the situation with X not compact are added.

0. Introduction

For a topological space X, let $C(X,\mathbb{C})$ be the ring of continuous complex valued functions on X, $C(X,\mathbb{R})$ the ring of continuous real valued functions on X. $\mathrm{KU}^*(X)$ is the topological complex K-theory of X, $\mathrm{KO}^*(X)$ the topological real K-theory of X. The main goal of this paper is the following result:

Theorem 7.3. *If X is a compact space, k any positive integer, then there are natural isomorphisms:*

$$K_i(C(X,\mathbb{C}),\mathbb{Z}/k\mathbb{Z})=\mathrm{KU}^{-i}(X,\mathbb{Z}/k\mathbb{Z}),$$
$$K_i(C(X,\mathbb{R}),\mathbb{Z}/k\mathbb{Z})=\mathrm{KO}^{-i}(X,\mathbb{Z}/k\mathbb{Z}) \qquad \text{for all } i\geq 0.$$

In other words, taking finite coefficients, the negative complex or real topological K-theory of a compact space equals the algebraic K-theory of the ring of continuous complex or real valued functions on X.

A referee pointed out to me that Prosolov [14] obtained independently the same result. The special case for $X=*$ a point was proved by Suslin in [19].

The most important tool is the following result of Gabber [3,4], Suslin [19] and Gillet and Thomason [5]:

Theorem A. *Let R be a $\mathbb{Z}[1/k]$-algebra that is a Henselian ring with maximal ideal $\mathfrak{m}$. Then $\tilde{H}_*(\mathrm{GL}(R,\mathfrak{m}),\mathbb{Z}/k\mathbb{Z})=0$ ($\tilde{H}_*$ denotes reduced homology).*

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The structure of this paper is as follows.

In the first section we introduce certain notions, in particular that of a locally convex topological group G; for example for a compact space X and $R=C(X,\mathbb{C})$ or $R=C(X,\mathbb{R})$, $G=\mathrm{GL}_n(R)$ is a locally convex topological group. Sections 2 and 3 contain the development of the machinery to be used, the construction of the homotopy fiber of the map $BH^\delta \to BG$ for a dense subgroup H of a locally convex topological group G and of a chain homotopy construction, used throughout to show that certain simplicial sets have trivial homology with finite coefficients.

Section 4 is in some sense a warm-up, as an example it uses the methods mentioned to reprove a theorem of Milnor. Section 5 applies the machinery to prove vanishing theorems for the homology of the homotopy fiber of the map $B\mathrm{GL}(R)^\delta \to B\mathrm{GL}(R)$, where R is a function ring.

The next two sections formulate the consequences of these vanishing theorems, Section 6 in homology, Section 7 in homotopy and K-theory, proving Theorem 7.3. Finally, Section 8 contains some remarks on function rings with a non-compact base space, extending the homology results of Section 6 and the K-theory results of Section 7.

This work grew out of my attempt to understand Suslin [19], so many of the ideas in this paper are elaborations of his. The approach taken here is somewhat more analytical than his algebraic approach, we are working mostly with germs of continuous maps instead of Henselizations of rings, or in other words, we are working with topological spaces and continuous maps as the primary structure, while [19] works with rings as the primary structure, using them to define the topology. The basic advantage is the easier control of small neighbourhoods than of large ring extensions.

1. Preliminaries

In this section, we want to introduce some notation, which will allow to present the later findings more clearly.

1.1. Homology of topological groups

For a topological group G, we denote by G^δ the same group with the discrete topology. We assign to G two simplicial sets:

(i) S_pG is the set of singular p-simplices of G,

(ii) S_pG^δ is the set of p-simplices in the bar construction on G^δ (i.e. $S_pG^\delta = (G^\delta)^p$).

We can apply the free abelian group functor $\mathbb{Z}$ to these simplicial sets and obtain chain complexes $\mathbb{Z}S_*G$, $C_*G^\delta := \mathbb{Z}S_*G^\delta$, which we can use to calculate the homology:

$H(\mathbb{Z}S_*G, \mathbb{Z}/k\mathbb{Z}) = H_*(G, \mathbb{Z}/k\mathbb{Z})$ is the homology of the *space* G,

$H(\mathbb{Z}S_*G^\delta, \mathbb{Z}/k\mathbb{Z}) = H_*(G^\delta, \mathbb{Z}/k\mathbb{Z})$ is the homology of the *discrete group* G.

1.2. The category of germs of topological spaces

We consider a space X with basepoint x_0. For the subsets of X we introduce the following equivalence relation: S_1 and S_2 are equivalent if and only if there is a neighbourhood U of x_0 such that $S_1 \cap U = S_2 \cap U$. The equivalence class of S is denoted $\underline{S}$ and called the germ of S at x_0. We will mostly consider the class $\underline{X}$.

Although $\underline{X}$ is not a topological space, we can define continuous maps: A map $f: \underline{X} \to Y$ is represented by a continuous map $f: U \to Y$ for some neighbourhood U of x_0, and two maps f and g are equivalent if there is a neighbourhood V of x_0 such that f and g are defined on V and $f|_V = g|_V$. The equivalence class of f is denoted by $\underline{f}$ or just by f, if no confusion can arise.

If f preserves the basepoint, then f maps some representative U of $\underline{X}$ into a representative V of $\underline{Y}$, and we write $\underline{f}: \underline{X} \to \underline{Y}$, similarly for germs of subsets $S \subset X$, $T \subset Y$, $\underline{g}: \underline{S} \to \underline{T}$. We have an obvious composition: $\underline{f}: \underline{X} \to \underline{Y}$, $\underline{h}: \underline{Y} \to \underline{Z}$, then $\underline{h} \circ \underline{f}: \underline{X} \to \underline{Z}$, and there are representatives f, h such that $\underline{h} \circ \underline{f} = \underline{h \circ f}$.

With this definition, we can talk about the category of germs of topological spaces and continuous maps. We will denote the set of continuous maps $\underline{f}: \underline{X} \to \underline{Y}$ and $\underline{f}: \underline{X} \to Y$ by $C(\underline{X}, \underline{Y})$ and $C(\underline{X}, Y)$ respectively.

There is an evaluation map $\mathrm{ev}: C(\underline{X}, Y) \to Y$, given by $\mathrm{ev}(\underline{f}) := f(x_0)$, split by the map $Y \to C(\underline{X}, Y)$ which assigns to $y \in Y$ the germ of the constant map with value y. Obviously $C(\underline{X}, \underline{Y}) = \mathrm{ev}^{-1}(y_0)$.

If G is a topological group, we take the identity element e as a basepoint. Then $C(\underline{X}, G)$ and $C(\underline{X}, \underline{G})$ inherit a group structure from G, and ev is a group homomorphism. In fact, we have a split short exact sequence $1 \to C(\underline{X}, \underline{G}) \to C(\underline{X}, G) \to G \to 1$.

1.3. The general linear group

Let R be a ring, $\mathrm{GL}_n(R)$ be the general linear group of R, after adjoining a unit if necessary.

If A is a topological ring (e.g. $A = \mathbb{R}$, $A = \mathbb{C}$ or $A = \mathbb{Q}$), we take 0 as basepoint and consider $R := C(\underline{X}, A)$, the ring of germs of continuous A-valued functions at $x_0 \in X$. Then the evaluation map is a ring homomorphism with kernel $\mathfrak{m} := C(\underline{X}, \underline{A}) \lhd R$ an ideal. For the general linear group, we have $\mathrm{GL}_n(R) = \mathrm{GL}_n(C(\underline{X}, A)) = C(\underline{X}, \mathrm{GL}_n(A))$, and since the short exact sequence $0 \to \mathfrak{m} \to R \to A \to 0$ splits, we have a short exact sequence $1 \to \mathrm{GL}_n(\mathfrak{m}) \to \mathrm{GL}_n(R) \to \mathrm{GL}_n(A) \to 1$ with $\mathrm{GL}_n(\mathfrak{m}) = \mathrm{GL}_n(R, \mathfrak{m})$ the congruence subgroup, given as $\mathrm{GL}_n(\mathfrak{m}) = \mathrm{GL}_n(C(\underline{X}, \underline{A})) = C(\underline{X}, \underline{\mathrm{GL}_n(A)})$.

1.4. Locally convex groups

If $A = \mathbb{R}$, then $\mathrm{GL}_n(A)$ is a Lie group and as such has arbitrarily small geodesically convex neighbourhoods U of the identity (cf. [7, § 5.2]). We give a short definition that extracts the essential feature of this notion for our purposes.

We start with a topological space Y and consider the pathspace $C(I, Y)$ with the map $\pi : C(I, Y) \to Y \times Y$, $p \mapsto (p(0), p(1))$. We call a section $s : Q \to C(I, Y)$ over a neighbourhood Q of the diagonal in $Y \times Y$ a *local convexity* of Y if it maps the diagonal to the constant paths. This means that for every pair $(x, y) \in Q$ we have a unique path $s(x, y)$ joining x and y, this path depends continuously on x and y and is the constant path at x if $y = x$. We will call such paths *geodesics*.

A set $U \subset Y$ with $U \times U \subset Q$ is called *convex* (relative to s) if $s(U \times U) \subset C(I, U)$, and Y is called *locally convex* (relative to s), if every point has arbitrarily small convex neighbourhoods. (This differs from the standard notion of 'locally convex', but the words describe precisely what we mean.) The following properties are immediate:

Lemma 1.1. *A convex set is contractible.* □

Lemma 1.2. *The intersection of convex sets is convex or empty.* □

For a topological group G, $g \in G$ acts on $f \in C(I, G)$ by $f \mapsto g \cdot f$. We want the convex sets of a group to be invariant under this action. This leads to the following:

Definition 1.3. A topological group G with a local convexity s is called a *locally convex* group, if G is locally convex as a topological space, $G \cdot Q \subset Q$ and $s(gx, gy) = g \cdot s(x, y)$ for all $(x, y) \in Q$ and all $g \in G$.

Examples. (i) Every open subset of an $\mathbb{R}$-vectorspace is locally convex with s given by straight lines.

(ii) Let $A = \mathbb{R}$ or $A = \mathbb{C}$ then $\mathrm{GL}_n(A)$ is a locally convex group: choose a ball $U = \{x \in M_n(A): |x - 1| \leq \varepsilon\}$ such that $U \subset \mathrm{GL}_n(A)$. U is convex in the standard sense. Let $Q := \{(x, y) \in \mathrm{GL}_n(A) \times \mathrm{GL}_n(A): x^{-1} \cdot y \in U\}$ and define for $(x, y) \in Q$ the line connecting x and y, $s(x, y)(t) := x + t \cdot (y - x)$, to be the convexity. Then $s(x, y)(t) \in \mathrm{GL}_n(A)$ for every t. This shows that $s : Q \to C(I, \mathrm{GL}_n(A))$ is a local convexity and $\mathrm{GL}_n(A)$ is a locally convex space. Since lines are preserved under the group action, it is a locally convex group. $\mathrm{GL}_n(A)$ is locally convex with respect to another local convexity too, see the next example.

(iii) If G is a Lie group, every element g has a geodesically convex neighbourhood U_g (see [7]). We define $Q := \bigcup_{g \in G} U_g \times U_g$, then for $(x, y) \in Q$ there is a g such that $(x, y) \in U_g \times U_g$, and $s(x, y)$ is the unique geodesic joining x and y, parametrized by arclength and normed to 1. The convex sets in our sense then are the geodesically convex sets of g contained in some U_g.

Proposition 1.4. *If G is a locally convex and locally compact group, X a compact topological space, then $C(X, G)$ is locally convex in the compact-open topology.*

Proof. For a convex neighbourhood U in G, define $\varphi : C(X, U) \times C(X, U) \to C(I, C(X, U))$ by $(f, g) \mapsto \tilde{\varphi} \circ (f, g)$, where $\tilde{\varphi}$ denotes the unique geodesic joining $f(x)$ and $g(x)$ for every $x \in X$. □

2. The construction of the homotopy fiber

In this section, we let G be a locally convex topological group. We choose a convex neighbourhood U of $1 \in G$. We start with a well known lemma, which discribes the weak homotopy type of a space, using an appropriate cover. A proof is contained in the proof of Proposition 4.1 in [19].

Lemma 2.1. *Let X be a topological space, $\mathfrak{U} = \{U_i : i \in I\}$ a collection of subsets of X whose interiors cover X and which has the property*

(gc) $\qquad U_\alpha = \bigcap_{i \in \alpha} U_i$ *is empty or contractible for all* $\alpha \in I^{n+1}$, $n \in \mathbb{N}$.

We call $\mathfrak{U}$ a good cover *of X.*

Then X is weakly homotopy equivalent to the geometric realization of the nerve of $\mathfrak{U}$, $N_*\mathfrak{U} := \{\alpha \in I^{n+1} : U_\alpha \neq \emptyset,\ n \in \mathbb{N}\}$. □

Now let H be a dense subgroup of G, we consider the underlying discrete group H^δ ($H = G$ is possible) and apply the lemma to obtain a description of the homotopy fiber of the map $BH^\delta \to BG$. We consider the fibration which we obtain from the usual fibration $G \to EG \to BG$ by first applying the singular simplex functor S_* and then the geometric realization functor $|-|$: $|S_*G| \to |S_*EG| \to |S_*BG|$.

We have the following cover of G: $\mathfrak{U} := \{h \cdot U : h \in H^\delta\}$, where U is any convex neighbourhood of $e \in G$; this is a cover of G since H is dense, and is a good cover by Lemma 1.2 and the definition of a locally convex group. The above lemma shows $|S_*G| \cong |N_*\mathfrak{U}|$. The equivalences are induced by H^δ-equivariant maps, so we get weak homotopy equivalences:

$$|S_*G|/H^\delta \xrightarrow{\cong} |N_*\mathfrak{U}|/H^\delta \xrightarrow{\cong} |N_*\mathfrak{U}/H^\delta|.$$

On the other hand, H^δ operates fiberwise on the fibration $|S_*G| \to |S_*EG| \to |S_*BG|$, freely and proper discontinuously on $|S_*G|$ and $|S_*EG|$, so we obtain a fibration

$$|S_*G|/H^\delta \cong |N_*\mathfrak{U}/H^\delta| \to |S_*EG|/H^\delta \to |S_*BG|.$$

Note that $EG \cong *$, so $|S_*EG| \cong EG \cong *$ and $|S_*EG|/H^\delta \cong BH^\delta$. We define $N_q U^\delta(H) := N_q\mathfrak{U}/H^\delta$. An element $h \in H$ operates on the right on $(h_0, \dots, h_q) \in N_q\mathfrak{U}$, and we choose the bar representation for the quotient:

$$[h_1|\dots|h_i] := (1, h_1, h_1h_2, \dots, h_1h_2\cdots h_i).$$

With this definition,

$$N_q U^\delta(H) = \left\{ [h_1|\dots|h_q] : h_i \in H,\ \bigcap_{i=0}^{q} h_1h_2\cdots h_i \cdot U \neq \emptyset \right\}$$

with the usual face and degeneracy operators (the empty product being defined as 1); this is a simplicial subset of S_qH^δ. The construction is independent of the particular choice of U in the sense that if we take a convex $V \subset G$ with $1 \in V \subset U$, then the inclusion $V \to U$ defines a homotopy equivalence $|N_* V^\delta(H)| \to |N_* U^\delta(H)|$. We summarize the di ussion.

Proposition 2.2. *Let G be a locally convex group. For every convex neighborhood U of $1 \in G$ and every dense subgroup H of G, we obtain a fibration up to homotopy:*

$$|N_* U^\delta(H)| \to |S_* H^\delta| \cong BH^\delta \to BG,$$

where the first map is induced by the inclusion. The homotopy fiber is independent of the particular choice of U. □

Definition 2.3. The homotopy fiber of the map $BH^\delta \to BG$ is denoted as $B\mathfrak{h}(G)$ or just $B\mathfrak{h}$ if G is clear from the context, the fiber of $BG^\delta \to BG$ is denoted as $B\mathfrak{g}$.

Remarks. (i) Proposition 2.2 is the same as Lemma 4.1 in [19] and Lemma 1 in [14].
(ii) If G is a Lie group, U an ε-ball around e, then $B\mathfrak{g}$ is the same as BG_ε in [19] and BG_U in [14].

3. The universal chain homotopy construction

We begin this section with some remarks from homological algebra. A double complex is a $\mathbb{Z}$-bigraded abelian group C_q^p with boundary and coboundary maps $d: C_q^p \to C_{q-1}^p$, $\delta: C_q^p \to C_q^{p+1}$, such that $d \circ \delta = \delta \circ d$. The associated total complex is $C^n := \prod_{p-q=n} C_q^p$ for $n \in \mathbb{Z}$ with $\bar{\delta} := (d - (-1)^n \delta): C^n \to C^{n+1}$. In this complex $y = (y_p) \in C^n$, $y_p \in C_p^{p+n}$, is a cycle if and only if $dy_p = (-1)^n \delta y_{p-1}$ for all $p \in \mathbb{Z}$. In the following, we will only consider double complexes with $C_q^p = 0$ if $p < 0$ or $q < 0$.

We call a double complex (C_q^p, d, δ) d-exact, if the sequence $C_{q+1}^p \to C_q^p \to C_{q-1}^p$ is exact for all p and all $q > 0$. The following is a standard result from homological algebra:

Lemma 3.1. *If the double complex (C_q^p, d, δ) is d-exact, then every cycle $y = (y_p) \in C^n$, $y_p \in C_p^{p+n}$ with $y_0 = 0$ is a boundary.* □

Now we apply this result to the situation we need. First we have to define the appropriate double complex.

For topological groups G and H and every pair $(p, q) \in \mathbb{N}^2$ we consider the set $S_q^p = S_q^p(G, H) := C(\underline{G}^p, \underline{H}^q)$. The standard face and degeneracy operators are continuous basepoint preserving maps and thus induce operations on S_q^p:

$$\begin{aligned}
d_j &: S_q^p \to S_{q-1}^p, & d_j(\underline{f}) &:= \underline{d_j \circ f},\\
s_j &: S_q^p \to S_{q+1}^p, & s_j(\underline{f}) &:= \underline{s_j \circ f},\\
\delta_j &: S_q^{p-1} \to S_q^p, & \delta_j(\underline{f}) &:= \underline{f \circ d_j},\\
\sigma_j &: S_q^{p+1} \to S_q^p, & \sigma_j(\underline{f}) &:= \underline{f \circ s_j}.
\end{aligned}$$

Note that the elements of $\{d_j, s_j\}$ commute with the elements of $\{\delta_j, \sigma_j\}$. With this, $\{S_q^p; d_j, s_j\}$ is a simplicial set for every p, $\{S_q^p; \delta_j, \sigma_j\}$ is a cosimplicial set for every q, and we obtain a double complex $C_q^p(G, H) := \mathbb{Z}S_q^p(G, H)$, the free abelian group generated by S_q^p.

For every fixed $p \in \mathbb{N}$, the homology of the complex $C_*^p(G, H)$ is the homology of the group $C(\underline{G}^p, \underline{H})$ with integer coefficients, so the double complex $C_q^p(G, H)$ is d-exact if and only if the homology of all these groups vanishes.

Now suppose we have a sequence of topological groups G_n, $n \in \mathbb{N}$, locally compact and Hausdorff, with $G_n \subset G_{n+1}$. We let $G_\infty := \operatorname{colim}\{G_n : n \in \mathbb{N}\}$ with the weak topology, and let $\mathbf{G}(p, n)$ be the group $C(\underline{G}_n^p, \underline{G}_\infty)$. For every couple (n, m) with $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{\infty\}$ we obtain a double complex $C_q^p(n, m) := C_q^p(G_n, G_m)$.

We assume that this double complex has the following property:

$(*)_{n,k}$ For $n \in \mathbb{N}$, the double complex $C_q^p(n, \infty) \otimes \mathbb{Z}/k\mathbb{Z}$ is d-exact, or equivalently, for $n \in \mathbb{N}$, $\tilde{H}_*(\mathbf{G}(p, n), \mathbb{Z}/k\mathbb{Z}) = 0$ for all p.

(An example of such a situation is given by $G_n := G$ for all n, G a solvable Lie group; this will be shown later.)

Fix some $n \in \mathbb{N}$. The germ of the identity map $\underline{f}_q : \underline{G}_n^q \to \underline{G}_n^q$, $q > 0$, together with $f_0 := 0$, induces an element $f = (\underline{f}_q)$ of degree 0 in the associated total complex $\bar{C}^*(n, \infty)$. f is a cycle in $\bar{C}^*(n, \infty)$: $df_1 = 0$, and for $p > 0$ for every $j = 0, \ldots, p$, $d_j f_p = \underline{d}_j = \bar{\delta}_j f_{p-1}$. If the condition $(*)_{n,k}$ holds, then $\underline{f}$ has to be a boundary $\operatorname{mod}(k)$, so there is $x = (x_p)$ such that $\bar{\delta} x \equiv \underline{f} \operatorname{mod}(k)$. Every x_p is a finite linear combination of elements of $S_p^{p-1}(n, \infty)$, and since G_n is locally compact Hausdorff, we can find a number $m \geq n$ such that $x_p \in C_p^{p-1}(n, m)$.

Corollary 3.2 (cf. [19]). *Let $\underline{f}_p : \underline{G}_n^p \to \underline{G}_n^p$ be the germ of the inclusion map, $l \in \mathbb{N}$. If the assumption $(*)_{n,k}$ holds, then there exists an $m \in \mathbb{N}$, $m \geq n$, and elements $x_p \in C_p^{p-1}(n, m)$ such that $\underline{f}_p \equiv dx_{p+1} + \delta x_p \operatorname{mod}(k)$ in $C_p^p(n, m)$ for all $p \leq l$.* □

We want to describe this result in terms of maps. Fix some $l \in \mathbb{N}$, then there is a neighbourhood U of $e \in G_n$ such that all x_p with $p \leq l$ are represented as linear combinations of continuous maps $x_p^i : U^{p-1} \to G_m^p$. Thus x_p induces a map $x_p : \mathbb{Z}U^{p-1} \to \mathbb{Z}V^p$ for some open neighbourhood of $e \in G_n$, and we have a diagram

$$\begin{tikzcd}
 & \mathbb{Z}V^{p+1} \arrow[d, "d"] \\
\mathbb{Z}U^p \arrow[ur, "x_{p+1}"] \arrow[r, "\mathrm{inc}"] \arrow[d, "d"'] & \mathbb{Z}V^p \\
\mathbb{Z}U^{p-1} \arrow[ur, "x_p"'] &
\end{tikzcd}$$

with $\mathrm{inc} \equiv d \circ x_{p+1} + x_p \circ d \bmod(k)$. This will be referred to as the universal chain homotopy. We formulate this as a proposition.

Proposition 3.3 (Universal chain homotopy construction). *Let G_m be a sequence of groups such that $(*)_{n,k}$ holds for some $n \in \mathbb{N}$. For every $l \in \mathbb{N}$ there is an $m \geq n$ with the following property:*

For every neighbourhood V of $e \in G_m$ there is a neighbourhood U of $e \in G_n$, with $U \subset V$ and morphisms $x_p : \mathbb{Z}U^{p-1} \to \mathbb{Z}V^p$, such that for the inclusion $\mathrm{inc} : U^p \to V^p$ *the congruence* $\mathrm{inc} \equiv d \circ x_{p+1} + x_p \circ d \bmod(k)$ *holds for all $0 < p \leq l$, whenever the terms on the right hand side are defined.* □

4. Example: Solvable Lie groups

In [12], Milnor proves that for a solvable Lie group G, the natural map $BG^\delta \to BG$ induces an isomorphism $H_*(BG^\delta, \mathbb{Z}/k\mathbb{Z}) \to H_*(BG, \mathbb{Z}/k\mathbb{Z})$. We want to reprove this result to show how our techniques relate to this question.

We start from the fact that for a uniquely k-divisible abelian group A, $\tilde{H}_*(A^\delta, \mathbb{Z}/k\mathbb{Z}) = 0$ (see e.g. [13, Lemma 3.13]). Now let X be any topological space, G a Lie group with Lie algebra $\mathfrak{g}$, then $C(\underline{X}, \underline{G})$ and $C(\underline{X}, \underline{\mathfrak{g}})$ inherit group structures. The exponential map $\exp : \mathfrak{g} \to G$ gives a diffeomorphism of a neighbourhood $0 \in U \subset \mathfrak{g}$ to a neighbourhood $e \in V \subset G$, so establishes a homeomorphism $\underline{\mathfrak{g}} \to \underline{G}$ and a bijection $C(\underline{X}, \underline{\mathfrak{g}}) \to C(\underline{X}, \underline{G})$, which is an isomorphism of groups if and only if $\mathfrak{g}$ (or equivalently the component of the identity of G) is abelian.

Lemma 4.1. *Let G be an abelian Lie group. Then $\tilde{H}_*(C(\underline{X}, \underline{G})^\delta, \mathbb{Z}/k\mathbb{Z}) = 0$ for every $k \in \mathbb{Z}$.*

Proof. Since G is abelian, $C(\underline{X}, \underline{G})$ is isomorphic to $C(\underline{X}, \underline{\mathfrak{g}})$ as an abelian group; and $C(\underline{X}, \underline{\mathfrak{g}})$ is a real vectorspace, so is uniquely divisible. □

For the Lie algebra $\mathfrak{g}$, the derived series is given by $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}_{i-1}]$. If there is an i such that $\mathfrak{g}_i = 0$, then $\mathfrak{g}$ and G are called solvable. Since every short exact sequence of Lie algebras $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{a} \to 0$ splits as a sequence of vectorspaces, the induced sequence $0 \to C(\underline{X}, \underline{\mathfrak{a}}) \to C(\underline{X}, \underline{\mathfrak{g}}) \to C(\underline{X}, \underline{\mathfrak{g}/\mathfrak{a}}) \to 0$ is exact, and if A, G are connected Lie groups with respective Lie algebras $\mathfrak{a}$ and $\mathfrak{g}$, then A is normal in G and $\mathfrak{g}/\mathfrak{a}$ is the Lie algebra of G/A. The sequence $0 \to \underline{A} \to \underline{G} \to \underline{G/A} \to 0$ splits as a sequence of germs of spaces, so $0 \to C(\underline{X}, \underline{A}) \to C(\underline{X}, \underline{G}) \to C(\underline{X}, \underline{G/A}) \to 0$ is a short exact sequence of groups.

Proposition 4.2. *Let G be a connected solvable Lie group. Then $\tilde{H}_*(C(\underline{X}, \underline{G})^\delta, \mathbb{Z}/k\mathbb{Z}) = 0$ for every $k \in \mathbb{Z}$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G, since G is solvable, there is an l such that $\mathfrak{g}_l=0$, so $\mathfrak{g}_{l-1}$ is abelian. For every i let G_i be a connected Lie group with Lie algebra $\mathfrak{g}_i$. Then G_{l-1} is abelian, so $H_j(C(\underline{X},\underline{G}_{l-1})^\delta,\mathbb{Z}/k\mathbb{Z})=0$ for $j>0$ and equal to $\mathbb{Z}/k\mathbb{Z}$ for $j=0$. Now suppose we have shown $\tilde{H}_*(C(\underline{X},\underline{G}_i)^\delta,\mathbb{Z}/k\mathbb{Z})=0$ for all $i>j$, then there is an exact sequence $0\to G_{j+1}\to G_j\to A\to 0$ with A an abelian Lie group. Now the Hochschild–Serre spectral sequence has the E^2-term

$$E^2_{pq}=H_p(C(\underline{X},\underline{A})^\delta,\ H_q(C(\underline{X},\underline{G}_{j+1})^\delta,\mathbb{Z}/k\mathbb{Z}))\ \Rightarrow\ H_{p+q}(C(\underline{X},\underline{G}_j)^\delta,\mathbb{Z}/k\mathbb{Z}).$$

By induction we know $\tilde{H}_q(C(\underline{X},\underline{G}_{j+1})^\delta,\mathbb{Z}/k\mathbb{Z})=0$, and $\tilde{H}_p(C(\underline{X},\underline{A})^\delta,\mathbb{Z}/k\mathbb{Z})=0$ by Lemma 4.1, so the spectral sequence collapses and shows $\tilde{H}_{p+q}(C(\underline{X},\underline{G}_j)^\delta,\mathbb{Z}/k\mathbb{Z})=0$. □

Proposition 4.3 (Milnor [12]). *If the component of the identity of G is solvable, then the map $G^\delta\to G$ induces an isomorphism in homology with finite coefficients*:

$$H_*(G^\delta,\mathbb{Z}/k\mathbb{Z})=H_*(BG^\delta,\mathbb{Z}/k\mathbb{Z})\xrightarrow{\cong}H_*(BG,\mathbb{Z}/k\mathbb{Z}).$$

Proof. (i) Since $\tilde{H}_*(C(\underline{X},\underline{G}_0)^\delta,\mathbb{Z}/k\mathbb{Z})=0$ for any space X, in particular for $X=G^n$, we can use Corollary 3.2 to construct a chain homotopy $x_p:\mathbb{Z}U^{p-1}\to\mathbb{Z}V^p$ with $\mathrm{id}\equiv d\circ x_{p+1}+x_p\circ d \bmod(k)$ (we let $G_n=G$ for all n).

(ii) Since G is locally convex, the homotopy fiber $B\mathfrak{g}$ of $BG^\delta\to BG$ is given as the geometric realization of N_*U^δ, with

$$N_pU^\delta=\left\{[g_1,\dots,g_p]\in G^p\colon \bigcap_{1=0}^{p} g_1\cdot\dots\cdot g_i\cdot U\neq\emptyset\right\}$$

for any convex neighbourhood U of $e\in G$. For a fixed l we can choose representatives $x_p=\sum a_ix_p^i$, $p\le l$, and find a convex neighbourhood U of e such that all the summands are defined on U^p. We can further assume that all x_p^i map U^p into V^{p+1} for some neighbourhood V of $e\in G$ such that $(g_1,\dots,g_{p+1})\in V^{p+1}$ implies $[g_1,\dots,g_{p+1}]\in N_{p+1}W^\delta$, where W is a convex neighbourhood, $e\in W\subset G$. Now we choose a convex neighbourhood U' with $N_pU'\subset U^p$, then the composition of maps $\mathbb{Z}N_p(U')^\delta\to\mathbb{Z}U^p\to\mathbb{Z}V^{p+1}\to\mathbb{Z}N_{p+1}W^\delta$ defines a contracting chain homotopy for the inclusion $N_*(U')^\delta\to N_*W^\delta$. Therefore $\tilde{H}_*(B\mathfrak{g},\mathbb{Z}/k\mathbb{Z})=0$, which proves the claim via the Serre spectral sequence of the fibration $B\mathfrak{g}\to BG^\delta\to BG$. (This kind of construction, in particular the choices of U', U, V, and W, will be explained in more detail in the next section.) □

5. The homology of the homotopy fiber

If R is a local ring with maximal ideal $\mathfrak{m}$, then we denote by $\bar{a}$ the image of a in $R/\mathfrak{m}$ and by $\bar{f}$ the image of an element f of $R[T]$ in $R/\mathfrak{m}[T]$. R is called a Henselian ring, if it has the following property:

(H) For all monic $f \in R[T]$ and $x \in R/\mathfrak{m}$ with $\bar{f}(x)=0$, $\bar{f}'(x) \in (R/\mathfrak{m})^*$, there is an $a \in R$ with $\bar{a}=x$ and $f(a)=0$.

We now let A be $\mathbb{R}$ or $\mathbb{C}$, X a topological space, R the ring of germs of continuous A-valued functions, $R=C(\underline{X}, A)$. R is a Henselian ring with maximal ideal $\mathfrak{m}=C(\underline{X}, \underline{A})$ (cf. [17, Chapitre VII, § 4]) and we have $\mathrm{GL}_m(R, \mathfrak{m})=C(\underline{X}, \mathrm{GL}_m(\underline{A}))$ (cf. Subsection 1.2). We define $\mathbf{GL}(X):=\operatorname{colim} C(\underline{X}, \mathrm{GL}_m(\underline{A}))$, then $\mathrm{GL}(R, \mathfrak{m})=\mathbf{GL}(X)$. Theorem A implies that $\tilde{H}_*(\mathbf{GL}(X), \mathbb{Z}/k\mathbb{Z})=0$ for any topological space X and $k>0$. When we consider the sequence of topological groups $G_m:=\mathrm{GL}(m, \mathbb{C})$ or $G_m:=\mathrm{GL}(m, \mathbb{R})$, then letting $X:=(G_m)^p$ we see that for the sequence $(G_m)_{m \in \mathbb{N}}$ $(*)_{n,k}$ holds for all $n \in \mathbb{N}$ and $k>0$. This allows us to construct certain maps that will yield stably a contracting homotopy.

We choose a convex neighbourhood U of $e \in \mathrm{GL}_n(A)$, and consider the homotopy fiber of $B\mathrm{GL}_n(A)^\delta \to B\mathrm{GL}_n(A)$, given as the geometric realization of the simplicial set

$$N_q U^\delta(A) := \left\{ [b_1, \ldots, b_q] \in \mathrm{GL}_n(A) \colon \bigcap_{i=0}^{q} b_1 \cdot \cdots \cdot b_i \cdot U \neq \emptyset \right\}.$$

Let $[b_1, \ldots, b_q] \in N_q U^\delta(A)$. Since $b_1 \cdot \cdots \cdot b_{i-1} \cdot U \cap b_1 \cdot \cdots \cdot b_i \cdot U \neq \emptyset$, we see $U \cap b_i \cdot U \neq \emptyset$, so $b_i \in U \cdot U^{-1}$ for all i. Let $V:=U \cdot U^{-1}$, then $N_q U^\delta(A) \subset V^q$. On the other hand, for a fixed $l \in \mathbb{N}$ and any neighbourhood U of e, we can choose a neighbourhood $W \subset U$ of e such that $W^{(-j)}:=\{b_1^{-1} \cdot \cdots \cdot b_j^{-1} \colon b_i \in W\} \subset U$ for all $j \leq l$. Then for $j \leq l$, $b_i \in W$, we get $b_j^{-1} \cdot \cdots \cdot b_1^{-1} \in W^{(-j)} \subset U$, and therefore $e = b_1 \cdot \cdots \cdot b_j \cdot (b_j^{-1} \cdot \cdots \cdot b_1^{-1}) \in b_1 \cdot \cdots \cdot b_j \cdot V$ for all $j \leq l$. This shows $\bigcap_{i=0}^{q} b_1 \cdot \cdots \cdot b_i \cdot U \neq \emptyset$, and altogether we obtain $W^q \subset N_q U^\delta(A) \subset V^q$ for all $q \leq l$.

We now apply the universal homotopy construction: For a given n, we choose m and $U' \subset V'$ such that $e \in U' \subset \mathrm{GL}_n(A)$, $e \in V' \subset \mathrm{GL}_m(A)$ and x_p is defined for all $p \leq l$; $x_p \colon \mathbb{Z}U'^{p-1} \to \mathbb{Z}V'^p$. The above arguments show that using appropriate restrictions, without loss of generality, we can find convex neighbourhoods U and V to obtain a composition:

$$N_{p-1} U^\delta(A) \to \mathbb{Z}U'^{p-1} \to \mathbb{Z}V'^p \to N_p V^\delta(A).$$

Thus x_p defines a map $N_{p-1} U^\delta(A) \to N_p V^\delta(A)$, and since $\mathrm{inc}_p \equiv d \circ x_{p+1} + x_p \circ d$ $\bmod(k)$, we see that the inclusion $N_p U^\delta(A) \to N_p V^\delta(A)$ induces the trivial map $\tilde{H}(N_* U^\delta(A) \otimes \mathbb{Z}/k\mathbb{Z}) \to \tilde{H}(N_* V^\delta(A) \otimes \mathbb{Z}/k\mathbb{Z})$. We remark in passing that this implies immediately the following:

Proposition 5.1 (Suslin [19, 4.3]). *Let $A=\mathbb{R}$ or $A=\mathbb{C}$, $\mathrm{GL}_n(A)$ the general linear group of A, $B\mathfrak{g}_n(A)$ the homotopy fiber of the map $B\mathrm{GL}_n(A)^\delta \to B\mathrm{GL}_n(A)$ for all $n \in \mathbb{N}$. Then for every n and l in $\mathbb{N}$ there is an $m \in \mathbb{N}$ such that the natural inclusion of the homotopy fibers $B\mathfrak{g}_n(A) \to B\mathfrak{g}_m(A)$ induces the trivial map*

$$H_i(B\mathfrak{g}_n(A), \mathbb{Z}/k\mathbb{Z}) \to H_i(B\mathfrak{g}_m(A), \mathbb{Z}/k\mathbb{Z}) \quad \textit{for } 0<i\leq l. \qquad \square$$

We now use this to deal with $R=C(X,A)$, X compact, $A=\mathbb{R}$ or $A=\mathbb{C}$. For every convex neighbourhood U of e in $\mathrm{GL}_n(A)$, $C(X,U)$ is a convex neighbourhood of the identity function in R in the compact-open topology by Proposition 1.4. So we obtain a description of the homotopy fiber of $B\mathrm{GL}_n(R)^\delta \to B\mathrm{GL}_n(R)$ as the geometric realisation of

$$N_pC(X,U)^\delta := \left\{[f_1,\dots,f_p]\in \mathrm{GL}_n(R)^p:\ \bigcap_{i=0}^{p} f_1\cdot\cdots\cdot f_i\cdot C(X,U)\neq\emptyset\right\}.$$

We now use the convex neighbourhoods U and V and the chain homotopy (x_p) from above. The composition of maps yields the sequence

$$N_pC(X,U)^\delta \to \mathbb{Z}C(X,U')^p \to \mathbb{Z}C(X,V')^{p+1} \to N_{p+1}C(X,V)^\delta,$$

from which we conclude as above the following:

Proposition 5.2. *For every n and every l in $\mathbb{N}$ there is an $m\in\mathbb{N}$ such that the natural inclusion $B\mathfrak{g}_n(R)\to B\mathfrak{g}_m(R)$ induces the trivial map*

$$H_i(B\mathfrak{g}_n(R),\mathbb{Z}/k\mathbb{Z})\to H_i(B\mathfrak{g}_m(R),\mathbb{Z}/k\mathbb{Z})\quad \textit{for all } 0<i\leq l. \qquad \square$$

6. Homology results

In this section, we want to formulate some formal consequences of the last results. We will assume that R is a ring which has the property: For every n, l there is an m such that the inclusion of the homotopy fiber construction $B\mathfrak{g}_n(R)\to B\mathfrak{g}_m(R)$ induces the trivial map in reduced homology $\tilde{H}_p(-,\mathbb{Z}/k\mathbb{Z})$ for $0\leq p\leq l$.

We take the colimit over all n: $\mathrm{GL}(R):=\mathrm{colim}\{\mathrm{GL}_n(R):\ n\in\mathbb{N}\}$, $\mathrm{GL}(R)^\delta:=\mathrm{colim}\{\mathrm{GL}_n(R)^\delta:\ n\in\mathbb{N}\}$, and $B\mathfrak{g}(R):=\mathrm{colim}\{B\mathfrak{g}_n(R):\ n\in\mathbb{N}\}$ is the homotopy fibre in $B\mathfrak{g}(R)\to B\mathrm{GL}(R)^\delta\to B\mathrm{GL}(R)$.

Corollary 6.1. *$H_i(B\mathfrak{g}(R),\mathbb{Z}/k\mathbb{Z})=0$ for $i>0$, $H_0(B\mathfrak{g}(R),\mathbb{Z}/k\mathbb{Z})=\mathbb{Z}/k\mathbb{Z}$.*

Proof. $H_*(B\mathfrak{g}(R),\mathbb{Z}/k\mathbb{Z})$ is generated by the images of $H_i(B\mathfrak{g}_n(R),\mathbb{Z}/k\mathbb{Z})\to H_i(B\mathfrak{g}(R),\mathbb{Z}/k\mathbb{Z})$, which factor through the trivial map $H_i(B\mathfrak{g}_n(R),\mathbb{Z}/k\mathbb{Z})\to H_i(B\mathfrak{g}_m(R),\mathbb{Z}/k\mathbb{Z})\to H_i(B\mathfrak{g}(R),\mathbb{Z}/k\mathbb{Z})$ for some m.

It remains to be shown that $B\mathfrak{g}(R)$ is connected. It suffices to show that every $B\mathfrak{g}_n(R)$ is. Consider the long exact homotopy sequence of $B\mathfrak{g}_n(R)\to B\mathrm{GL}_n(R)^\delta\to B\mathrm{GL}_n(R)$:

$$\begin{array}{ccccccc}
\pi_1 B\mathrm{GL}_n(R)^\delta & \longrightarrow & \pi_1 B\mathrm{GL}_n(R) & \longrightarrow & \pi_0 B\mathfrak{g}_n(R) & \longrightarrow & \pi_0 B\mathrm{GL}_n(R)^\delta \\
\| & & \| & & & & \| \\
\mathrm{GL}_n(R)^\delta & \longrightarrow & \pi_0 \mathrm{GL}_n(R) & & & & \{*\}.
\end{array}$$

So the map $\pi_1 B\mathrm{GL}_n(R) \to \pi_0 B\mathfrak{g}_n(R)$ is surjective. The map $\pi_1 B\mathrm{GL}_n(R)^\delta \to \pi_1 B\mathrm{GL}_n(R)$ is the map $\mathrm{GL}_n(R)^\delta \to \pi_0 \mathrm{GL}_n(R)$ that takes an element to its connectivity component, so this is surjective too. Therefore $\pi_0 B\mathfrak{g}_n(R) = *$. $\square$

Remark. According to a conjecture due to Quillen and Friedlander, for $R = \mathbb{R}$, $\tilde{H}_*(B\mathfrak{g}_n(R), \mathbb{Z}/k\mathbb{Z})$ should be zero for all n (cf. [12; 16, p. 176]). Our input of Theorem A allows only the stable result, but an improvement of Theorem A to the unstable case would immediately lead to the desired result for $B\mathfrak{g}_n(R)$.

Corollary 6.2. $H_*(B\mathrm{GL}(R)^\delta, \mathbb{Z}/k\mathbb{Z}) \cong H_*(B\mathrm{GL}(R), \mathbb{Z}/k\mathbb{Z})$.

Proof. Follows from the Serre spectral sequence of the fibration $B\mathfrak{g}(R) \to B\mathrm{GL}(R)^\delta \to B\mathrm{GL}(R)$ (cf. [25, Theorem XIII, 7.15, p. 653]). $\square$

We want to establish a similar result for the special linear group. We now assume that R is commutative, then for $f \in \mathrm{GL}_n(R)$ the determinant of f is well defined, and

$$s := \begin{pmatrix} f & 0 \\ 0 & (\det f)^{-1} \end{pmatrix}$$

gives a splitting in the following sense:

$$\begin{tikzcd} \mathrm{SL}_n(R) \arrow[r] \arrow[dr] & \mathrm{GL}_n(R) \arrow[d, "s"] \\ & \mathrm{SL}_{n+1}(R) \end{tikzcd}$$
commutes.

This defines maps on the classifying spaces and thus on the homotopy fibre:

$$\begin{tikzcd} B\mathfrak{g}_n(R) \arrow[r] \arrow[d] & B\mathrm{GL}_n(R)^\delta \arrow[r] \arrow[d] & B\mathrm{GL}_n(\bar{R}) \arrow[d] \\ B\mathfrak{s}_{n+1}(R) \arrow[r] & B\mathrm{SL}_{n+1}(R)^\delta \arrow[r] & B\mathrm{SL}_{n+1}(R). \end{tikzcd}$$

We fix some $l \in N$ and choose $m \geq n$ such that

$$H_i(B\mathfrak{g}_n(R), \mathbb{Z}/k\mathbb{Z}) \to H_i(B\mathfrak{g}_m(R), \mathbb{Z}/k\mathbb{Z})$$

is trivial for $0 < i \leq l$. Since the inclusion factors: $\mathrm{SL}_n(R) \to \mathrm{GL}_n(R) \to \mathrm{GL}_m(R) \xrightarrow{s} \mathrm{SL}_{m+1}(R)$, so does $B\mathfrak{s}_n(R) \to B\mathfrak{g}_n(R) \to B\mathfrak{g}_m(R) \to B\mathfrak{s}_{m+1}(R)$, and we obtain the following:

Proposition 6.3. *For every n and every l, there is a $m > n$ such that the natural inclusion $B\mathfrak{s}_n(R) \to B\mathfrak{s}_m(R)$ induces the trivial map in homology with $\mathbb{Z}/k\mathbb{Z}$-coefficients in the dimensions $0 < i \leq l$.* $\square$

Corollary 6.4.

$$H_i(B\mathfrak{g}(R), \mathbb{Z}/k\mathbb{Z}) = 0 \quad \textit{for } i > 0,$$

$$H_0(B\mathfrak{g}(R), \mathbb{Z}/k\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z},$$

$$H_*(B\mathrm{SL}(R)^\delta, \mathbb{Z}/k\mathbb{Z}) \cong H_*(B\mathrm{SL}(R), \mathbb{Z}/k\mathbb{Z}).$$

Proof. The same as above. □

7. Homotopy results

We want to transfer our results to homotopy. There is a functor from the category of topological rings into the category of infinite loop spaces, that assigns to a ring R the Quillen construction $B\mathrm{GL}(R)^+$ (cf. [10]). For our purposes, this shows that the map $R^\delta \to R$ induces an infinite loop map $j: B\mathrm{GL}(R^\delta)^+ \to B\mathrm{GL}(R)^+$. Again we let $A = \mathbb{R}$ or $A = \mathbb{C}$, and R the ring of A-valued functions on a compact space X with compact-open topology.

To see that $\pi_1(B\mathrm{GL}(R)) = \pi_0\mathrm{GL}(R)$ is abelian, note that the commutator group is $[\mathrm{GL}(R), \mathrm{GL}(R)] = E(R)$, and $E(R)$ is an open path component of $\mathrm{GL}(R)$ [11, § 7]. Thus we conclude $B\mathrm{GL}(R) = B\mathrm{GL}(R)^+$.

Since the maximal perfect subgroup of $\pi_1 B\mathrm{GL}(R)$ is trivial, applying the $+$-construction to the fibration $B\mathfrak{g}(R) \to B\mathrm{GL}(R^\delta) \to B\mathrm{GL}(R)$ yields again a fibration $B\mathfrak{g}(R)^+ \to B\mathrm{GL}(R^\delta)^+ \xrightarrow{j} B\mathrm{GL}(R)^+ = B\mathrm{GL}(R)$ (see [2, Theorem 6.4]). We denote the homotopy fiber of j by $Fj \cong B\mathfrak{g}(R)^+$. The $+$-construction leaves homology unchanged, so we have by Corollary 6.1 $\tilde{H}_*(Fj, \mathbb{Z}/k\mathbb{Z}) = 0$. We want to show that j induces an isomorphism in homotopy with finite coefficients.

First we need a lemma. Let $f: S \to T$ be a map of 1-connected spaces, with homotopy fiber F, $\Omega F \to \Omega S \xrightarrow{\Omega f} \Omega T$ the fibration induced by the loop space functor.

Lemma 7.1. *If $\tilde{H}_*(\Omega F, D) = 0$ for an abelian group D and F is 1-connected, then $\tilde{H}_*(F, D) = 0$ and $f_*: H_*(S, D) \to H_*(T, D)$ is an isomorphism.*

Proof. We use the following proposition (cf. [25, Theorem XIII, 7.15, p. 653]): Let D be an abelian group, $g: Y \to Z$ a D-orientable map with fiber X (i.e. $\pi_1(Z)$ acts trivially on $H_*(x, D)$). Then X is acyclic over D if and only if $g_*: H_*(Y, D) \to H_*(Z, D)$ is an isomorphism.

We know that $\tilde{H}_*(\Omega F, D) = 0$, ΩF is also the fiber of $PF \to F$, this fibration is orientable since $\pi_1(F) = 0$, so $H_*(PF, D) \to H_*(F, D)$ is an isomorphism. But $PF \cong *$, so $\tilde{H}_*(F, D) = 0$, and $f_*: H_*(S, D) \to H_*(T, D)$ is an isomorphism. □

Now consider the infinite loop map j. There are 1-connected spaces S_1 and T_1 with $\Omega S_1 = B\mathrm{GL}(R^\delta)^+$ and $\Omega T_1 = B\mathrm{GL}(R)$, and a map $j_1: S_1 \to T_1$ such that $\Omega j_1 = j$.

We choose a homotopy fiber Fj_1 of j_1 such that $\Omega Fj_1 \cong Fj$. Fj is connected and $\mathbb{Z}/k\mathbb{Z}$-acyclic by Corollary 6.2, therefore Fj_1 is 1-connected and Lemma 7.1 implies: j_1 induces an isomorphism in homology with finite coefficients. But S_1 and T_1 are simply connected, therefore j_1 induces an isomorphism in homotopy with finite coefficients $\pi_i(S_1, \mathbb{Z}/k\mathbb{Z}) \cong \pi_i(T_1, \mathbb{Z}/k\mathbb{Z})$ for $i \geq 2$ (cf. [13, Corollary 3.10]). With this we obtain

$$\begin{aligned}\pi_i(B\mathrm{GL}(R^\delta)^+, \mathbb{Z}/k\mathbb{Z}) &\cong \pi_{i+1}(S_1, \mathbb{Z}/k\mathbb{Z})\\ &\cong \pi_{i+1}(T_1, \mathbb{Z}/k\mathbb{Z})\\ &\cong \pi_i(B\mathrm{GL}(R), \mathbb{Z}/k\mathbb{Z})\end{aligned}$$

for $i \geq 2$. For $i = 1$, we have by definition:

$$\begin{aligned}\pi_1(B\mathrm{GL}(R^\delta)^+, \mathbb{Z}/k\mathbb{Z}) &= \pi_1(B\mathrm{GL}(R^\delta)^+) \otimes \mathbb{Z}/k\mathbb{Z}\\ &= H_1(B\mathrm{GL}(R^\delta)^+, \mathbb{Z}/k\mathbb{Z}) \quad \text{(by the Hurewicz isomorphism)}\\ &= H_1(B\mathrm{GL}(R^\delta), \mathbb{Z}/k\mathbb{Z})\end{aligned}$$

and similarly $\pi_1(B\mathrm{GL}(R), \mathbb{Z}/k\mathbb{Z}) = H_1(B\mathrm{GL}(R), \mathbb{Z}/k\mathbb{Z})$. This yields together the following:

Proposition 7.2. *$\pi_i(B\mathrm{GL}(R^\delta)^+, \mathbb{Z}/k\mathbb{Z}) \cong \pi_i(B\mathrm{GL}(R), \mathbb{Z}/k\mathbb{Z})$ for $i > 0$.* $\square$

We will describe this result in the language of the algebraic K-theory of the ring $R = C(X, A)$, $A = \mathbb{R}$ or $R = \mathbb{C}$. We let K stand for KO if $A = \mathbb{R}$ and for KU if $A = \mathbb{C}$. Then according to Swan, $K_0 R = K^0 X$ [20], and Milnor proved the existence of exact sequences $K_2 R \to K^{-2} X \to R \xrightarrow{\exp} K_1 R \to K^{-1} X \to 0 \to K_0 R \to K^0 X \to 0$ [11, §7], which shows that $K_1 R$ differs from $K^{-1} X$ only by divisible elements. Although this result is not used in the sequel, it provides motivation for conjecturing $K_i(R, \mathbb{Z}/k\mathbb{Z}) \cong K^{-i}(X, \mathbb{Z}/k\mathbb{Z})$.

We begin by giving the definitions: for a ring R the K-theory with finite coefficients, $K_i(R, \mathbb{Z}/k\mathbb{Z})$ is defined as $\pi_{i+1}(BQ\mathscr{P}(R), \mathbb{Z}/k\mathbb{Z})$ for $i \geq 0$ (see e.g. [24]). Here $\mathscr{P}(R)$ is the category of finitely generated projective R-modules, and since $\Omega BQ\mathscr{P}(R) \cong K_0 R \times B\mathrm{GL}(R)^+$ [6], $\pi_1(BQ\mathscr{P}(R)) = \pi_0(K_0 R \times B\mathrm{GL}(R)^+) = K_0 R$ is abelian and therefore $K_0(R, \mathbb{Z}/k\mathbb{Z}) = K_0 R \otimes \mathbb{Z}/k\mathbb{Z}$ is well defined (this corresponds to taking a (-1)-connected spectrum for algebraic K-theory). We have $K_i(R, \mathbb{Z}/k\mathbb{Z}) = \pi_i(B\mathrm{GL}(R^\delta)^+, \mathbb{Z}/k\mathbb{Z})$ for $i \geq 2$, $K_0(R, \mathbb{Z}/k\mathbb{Z}) = K_0 R \otimes \mathbb{Z}/k\mathbb{Z}$, and a short exact sequence (cf. [13, Proposition 1.4])

(i) $$0 \to \pi_1 B\mathrm{GL}(R^\delta)^+ \otimes \mathbb{Z}/k\mathbb{Z} \to K_1(R, \mathbb{Z}/k\mathbb{Z}) \to \mathrm{Tor}^{\mathbb{Z}}(K_0 R, \mathbb{Z}/k\mathbb{Z}) \to 0.$$

The (real or complex) topological K-theory of a compact space is given by $K^{-n}(X, \mathbb{Z}/k\mathbb{Z}) = [X^+ \wedge S^n(k), B\mathrm{GL}(A)]_*$ for $n \geq 2$, where $X^+ = X \cup \{*\}$ denotes the 'one point compactification', $S^n(k)$ the Moore space and $[\ldots]_*$ homotopy classes with fixed base point. To avoid K^1, we define $K^0(X,\mathbb{Z}/k\mathbb{Z}):=K^0X\otimes\mathbb{Z}/k\mathbb{Z}$, and we have an exact sequence (cf. [1])

$$\text{(ii)}\qquad 0\to K^{-1}(X)\otimes\mathbb{Z}/k\mathbb{Z}\to K^{-1}(X,\mathbb{Z}/k\mathbb{Z})\to\operatorname{Tor}^{\mathbb{Z}}(K^0(X),\mathbb{Z}/k\mathbb{Z})\to 0.$$

Now Proposition 5.2 shows

$$\begin{aligned}K_i(R,\mathbb{Z}/k\mathbb{Z})&=\pi_i(B\mathrm{GL}(R^\delta)^+,\mathbb{Z}/k\mathbb{Z})=\pi_i(B\mathrm{GL}(R),\mathbb{Z}/k\mathbb{Z})\\&=[S^i(k),B\mathrm{GL}(R)]_*=[S^i(k),C(X,B\mathrm{GL}(A)]_*\\&=[X^+\wedge S^i(k),B\mathrm{GL}(A)]_*=K^{-i}(X,\mathbb{Z}/k\mathbb{Z})\quad\text{for } i\geq 2.\end{aligned}$$

For $i=0$, Quillen showed that $\pi_1(BQ\mathscr{P}(R))$ equals the classical Grothendieck group K_0R [15], and by the Serre–Swan theorem, $K_0R=K^0X$. We saw further:

$$\begin{aligned}\pi_1B\mathrm{GL}(R^\delta)^+\otimes\mathbb{Z}/k\mathbb{Z}&=\pi_1(B\mathrm{GL}(R^\delta)^+,\mathbb{Z}/k\mathbb{Z})\cong\pi_1(B\mathrm{GL}(R),\mathbb{Z}/k\mathbb{Z})\\&=K^{-1}(X)\otimes\mathbb{Z}/k\mathbb{Z}.\end{aligned}$$

Therefore the first and the third terms in the exact sequences (i) and (ii) are naturally isomorphic, so the central terms by the 5-Lemma have to be naturally isomorphic too. We formulate now the main result.

Theorem 7.3. *For a compact space X, there are natural isomorphisms*

$$K_i(C(X,\mathbb{C}),\mathbb{Z}/k\mathbb{Z})\cong\mathrm{KU}^{-i}(X,\mathbb{Z}/k\mathbb{Z})$$

and

$$K_i(C(X,\mathbb{R}),\mathbb{Z}/k\mathbb{Z})\cong\mathrm{KO}^{-i}(X,\mathbb{Z}/k\mathbb{Z})$$

for all $i\geq 0$. □

8. Some remarks on noncompact spaces

When we try to apply the above theory to the ring of continuous A-valued functions $R:=C(X,A)$ on a noncompact space X, we run into certain difficulties. We assume that A admits a metric, then we can consider the uniform topology on R (denoted by subscript uf) or the compact-open topology (R_{co}). But R_{uf} is not a topological ring and $\mathrm{GL}(R_{\mathrm{uf}})$ not a topological group. On the other hand, $\mathrm{GL}(R_{\mathrm{co}})=\lim C(X,\mathrm{GL}_n(A))$, but $C(X,\mathrm{GL}_n(A))$ is in general not locally convex in the sense of Definition 1.3, so the universal homotopy construction does not apply.

An alternative approach is to consider a subring instead of the full ring $C(X,A)$. For simplicity we focus on $A=\mathbb{R}$ and assume X to be a locally compact Hausdorff space.

Let X be locally compact and define $\bar{X}=X\cup\{\infty\}$ to be the one-point compactification of X. Following Karoubi [8] we define $\mathrm{KO}^{-n}(X,C):=\operatorname{Ker}(K^{-n}(\bar{X},C)\to K^{-n}(\{\infty\},C))$ for $n\geq 0$. Recall that we are working (-1)-connected, so $\mathrm{KO}^0(X,C)=\mathrm{KO}^0X\otimes C$.

The ring of bounded functions, $R_b := C_b(X, \mathbb{R})$. We consider the Stone–Cech-compactification βX of X. By definition we have $C_b(X, \mathbb{R}) = C(\beta X, \mathbb{R})$, and Theorem 7.3 yields:

$$K_i(C_b(X, \mathbb{R}), \mathbb{Z}/k\mathbb{Z}) = \mathrm{KO}^{-i}(\beta X, \mathbb{Z}/k\mathbb{Z}) \quad \text{for } i \geq 0.$$

Since R_b is a Banach algebra, this follows from Prasolov [14] too. Vaserstein showed that $K_0(R_b)$ is the Grothendieck group of the category of vectorbundles of finite type on X [22] and $K_1(R_b) = C_b(X, \mathbb{R}^*) \times \lim\, [X, \mathrm{SO}_n(\mathbb{R})]$ [23].

The ring of functions vanishing at infinity, $R_0 := C_0(X, \mathbb{R})$. For the one-point compactification $\bar{X} = X \cup \{\infty\}$ of X we define as $\bar{R} := C(\bar{X}, \mathbb{R})$ the function ring on $\bar{X}$, then R_0 is an ideal in $\bar{R}$, and we have a split-exact sequence $0 \to R_0 \to \bar{R} \to \mathbb{R} \to 0$, where the map $\bar{R} \to \mathbb{R}$ is evaluation at ∞, and the splitting is given by the constant functions in $\bar{R}$.

We can consider the low-dimensional exact sequence in K-theory: $K_1(\bar{R}, R_0) \to K_1\bar{R} \to K_1\mathbb{R} \to K_0(\bar{R}, R_0) \to K_0\bar{R} \to K_0\mathbb{R} \to 0$, then the splitting shows that $0 \to K_0(\bar{R}, R_0) \to K_0\bar{R} \to K_0\mathbb{R} \to 0$ is exact. We get $K_0R_0 = K_0(\bar{R}, R_0) = \ker(K_0\bar{R} \to K_0\mathbb{R}) = \ker(K^0\bar{X} \to K^0\{\infty\})$. Since we know $K^0\bar{X} = K_0\bar{R}$ and $K^0\{\infty\} = K_0\mathbb{R}$, we conclude $K_0R_0 = K^0X$.

To describe the higher K-theory with finite coefficients, we observe that $\bar{R}$ is a $\mathbb{Z}[1/k]$-algebra with unit, so $K_*(R_0, \mathbb{Z}/k\mathbb{Z}) = K_*(\bar{R}, R_0; \mathbb{Z}/k\mathbb{Z})$ [24] and we get a long exact sequence in K-theory with finite coefficients, which splits into short exact sequences $0 \to K_*(R_0, \mathbb{Z}/k\mathbb{Z}) \to K_*(\bar{R}, \mathbb{Z}/k\mathbb{Z}) \to K_*(\mathbb{R}, \mathbb{Z}/k\mathbb{Z}) \to 0$. Therefore Theorem 7.3 applied to $\bar{X}$ and $\{\infty\}$, together with $K_0R_0 = \mathrm{KO}^0X$ yields the following:

Proposition 8.1. *There are natural isomorphisms*

$$K_i(C_0(X, \mathbb{R}), \mathbb{Z}/k\mathbb{Z}) \to \mathrm{KO}^{-i}(X, \mathbb{Z}/k\mathbb{Z}) \quad \textit{for all } i \geq 0. \qquad \square$$

(I am indebted to Bob Thomason for pointing this out to me.)

The ring of functions with compact support, $R_c := C_c(X, \mathbb{R})$. We now have a (non-split) exact sequence $0 \to R_c \to \bar{R} \to \tilde{R} \to 0$, where $\bar{X}$ and $\bar{R}$ as above and $\tilde{R} = \bar{R}/R_c = C(\underline{\bar{X}}, \mathbb{R})$ is the ring of germs of continuous $\mathbb{R}$-valued functions at ∞, $\tilde{R}$ is a local Henselian ring.

We consider the low dimensional exact sequence $K_1(\bar{R}, R_c) \to K_1\bar{R} \to K_1\tilde{R} \to K_0(\bar{R}, R_c) \to K_0\bar{R} \to K_0\tilde{R}$. $\tilde{R}$ local implies $K_1\tilde{R} = \tilde{R}^*$, $K_0\tilde{R} = \mathbb{Z} = K_0\mathbb{R}$. Furthermore, $K_1\bar{R} = \bar{R}^* \oplus [\bar{X}, \mathrm{SO}(\mathbb{R})]$ [11, Corollary 7.3], and the natural map $\bar{R}^* \to \tilde{R}^*$ is surjective by Tietze extension, so we see that $0 \to K_0(\bar{R}, R_c) \to K_0\bar{R} \to K_0\tilde{R} = K_0\mathbb{R}$ is exact, and $K_0R_c = K_0(\bar{R}, R_c) = K_0R_0 = K^0X$.

For higher K-theory, we have the long exact sequence

$$\cdots \to K_n(\bar{R}, R_c) \to K_n\bar{R} \to K_n\tilde{R} \to K_{n-1}(\bar{R}, R_c) \to \cdots \to K_0\bar{R} \to K_0\tilde{R}.$$

R_c has the following property: for every finite subset $S \subset R_c$, there is an element $a \in R_c$ such that $ab = b$ for all $b \in S$. According to Vaserstein [21, Theorem 17.1], this signifies that the relative K-theory $K_n(\bar{R}, R_c)$ is independent of the ring $\bar{R}$ and equals K_nR_c. With this we have a long exact sequence

$$\cdots \to K_nR_c \to K_n\bar{R} \to K_n\tilde{R} \to K_{n-1}R_c \to \cdots \to K_0\bar{R} \to K_0\tilde{R}.$$

If we use finite coefficients, then Weibel's result yields a long exact sequence

$$\cdots \to K_n(R_c, \mathbb{Z}/k\mathbb{Z}) \to K_n(\bar{R}, \mathbb{Z}/k\mathbb{Z}) \to K_n(\tilde{R}, \mathbb{Z}/k\mathbb{Z})$$
$$\to K_{n-1}(R_c, \mathbb{Z}/k\mathbb{Z}) \to \cdots \to K_0(\bar{R}, \mathbb{Z}/k\mathbb{Z}) \to K_0(\tilde{R}, \mathbb{Z}/k\mathbb{Z}).$$

By Theorem 7.3 we know $K_n(\bar{R}, \mathbb{Z}/k\mathbb{Z}) \cong \mathrm{KO}^{-n}(\bar{X}, \mathbb{Z}/k\mathbb{Z})$, and by Theorem A we know $K_n(\tilde{R}, \mathbb{Z}/k\mathbb{Z}) \cong K_n(\mathbb{R}, \mathbb{Z}/k\mathbb{Z})$, since $\tilde{R}$ is Henselian.

Now comparison with the long exact sequence above shows $K_n(R_c, \mathbb{Z}/k\mathbb{Z}) \cong K_n(R_0, \mathbb{Z}/k\mathbb{Z}) \cong \mathrm{KO}^{-n}(X, \mathbb{Z}/k\mathbb{Z})$. We obtained the following:

Theorem 8.2. *Let X be a locally compact space. Then there is a natural isomorphism $K_i(C_c(X, \mathbb{R}), \mathbb{Z}/k\mathbb{Z}) \to \mathrm{KO}^{-i}(X, \mathbb{Z}/k\mathbb{Z})$ for all $i \geq 0$.* □

We add some remarks about the homology of the infinite general linear groups. We can consider R_0 and R_c as subspaces of $\bar{R}$, they then carry the uniform topology. $\mathrm{GL}_n(R_c)$ is dense in $\mathrm{GL}_n(R_0)$ by Tietze extension applied to the normal space $\bar{X}$, and $\mathrm{GL}_n(R_0)$ is a locally convex subspace of $\mathrm{GL}_n(\bar{R})$, so we can apply the construction of the homotopy fiber in Proposition 2.2 to $\mathrm{GL}_n(R_c^\delta) \to \mathrm{GL}_n(R_0)$ or to $\mathrm{GL}_n(R_0^\delta) \to \mathrm{GL}_n(R_0)$. Let x stand for 0 or c, then we obtain $B\mathfrak{g}_n(R_x, R_0)$ as the geometric realization of the simplicial set which has in dimension q the simplices

$$N_qC(X, U)^\delta = \left\{ [f_1, \ldots, f_q] \in \mathrm{GL}_n(R_x^\delta)^q \colon \bigcap_{i=0}^{q} f_1 \cdot \cdots \cdot f_i \cdot C(X, U) \neq \emptyset \right\}.$$

The universal chain homotopy construction takes elements of $\mathrm{GL}_n(R_x)$ to elements of $\mathrm{GL}_m(R_x)$ for some $m \geq n$, so we see that for every $l \in \mathbb{N}$ we can find an $m \in \mathbb{N}$ such that the inclusion $B\mathfrak{g}_n(R_x, R_0) \to B\mathfrak{g}_m(R_x, R_0)$ induces the trivial map $\tilde{H}^*(B\mathfrak{g}_n(R_x, R_0), \mathbb{Z}/k\mathbb{Z}) \to \tilde{H}^*(B\mathfrak{g}_m(R_x, R_0), \mathbb{Z}/k\mathbb{Z})$. As above we conclude that the colimit $\tilde{H}^*(B\mathfrak{g}(R_x, R_0), \mathbb{Z}/k\mathbb{Z}) = 0$ and

$$\tilde{H}^*(\mathrm{GL}(R_c^\delta), \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\cong} \tilde{H}^*(\mathrm{GL}(R_0), \mathbb{Z}/k\mathbb{Z}) \xleftarrow{\cong} \tilde{H}^*(\mathrm{GL}(R_0^\delta), \mathbb{Z}/k\mathbb{Z}).$$

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